ON BANACH LATTICES OF OPERATORS

BY

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ABSTRACT

Let Λ_1 and Λ_2 be infinte-dimensional Banach lattices such that c_0 is not finitely representable in Λ_2. Then the bounded linear operators from Λ_1 to Λ_2 form a lattice if and only if Λ_1 is an abstract AL space.

In [2], Fremlin showed that a Banach lattice Λ is isomorphic to an AL-space (see, e.g. [1]) if every compact linear operator $T:\Lambda \rightarrow l_1$ is the difference of two positive operators. The purpose of this note is to prove the following extension of Fremlin's result: *If Λ_1 and Λ_2 are infinite dimensional B - lattices such that every operator in the uniform closure $\mathcal{F}(\Lambda_1, \Lambda_2)$ of the finite rank operators from Λ_1 to Λ_2 splits by $T = P - N$ with P and N positive, and if Λ_2 does not contain (uniformly) copies of l_∞^n, then Λ_1 is isomorphic (linear, order and norm) to an* AL-*space.* The basic technique is Fremlin's. The new parts of the proof are based on some inequalities due to Tzafriri [5]. This result is suggested by a well known result (see, e.g. [4, p. 174]) that every operator from an AL-space to a B-lattice not containing c_0 split (as above).

We now proceed to the proof of the main result.

Notice first that since $T \in \mathcal{F}(\Lambda_1, \Lambda_2)$ can be written as $T = P - N$, we may define $|T|$ on positive elements x of Λ_1 by the formula $|T|x = \vee\{Ty \mid -x \leqq y \leqq x\}$. In fact, for any $x \geqq 0$ in Λ_1, and any $|y| \leqq x$, $Ty = Ty^+ - Ty^- \leqq Py^+ + Ny^- \leqq Px + Nx$, so that $|T|x$ exists by the order completeness of Λ_2 [3]. It is also clear that $\|T\| \leqq \| |T| \|$ and $|T+S| \leqq |T| + |S|$. We wish to show that there exists $K \geqq 1$ such that $\| |T| \| \leqq K\|T\|$ for all $T \in \mathcal{F}(\Lambda_1, \Lambda_2)$. For this, it is sufficient to prove that the ball $\{T : \| |T| \| \leqq 1\}$ is

Received August 28, 1974

closed. This is obvious since this ball is the intersection of the family of all closed sets of the form $\{S \mid \|\vee_{i=1}^{n} Sy_i\| \leqq 1\}$ where $\vee_{i=1}^{n}|y_i| \leqq |x|$ for some $x, \|x\| \leqq 1$.

Next, according to Tzafriri, there is an $\alpha > 0$ such that for any n, there exist $e_1, \cdots, e_{2^n}$, non-negative, disjoint, normalized elements of Λ_2 such that the Rademacher elements over $(e_i)_{i=1}^{2^n}$

$$r_1 = \left\{\sum_{i=1}^{2^{n-1}} e_i - \sum_{i=2^{n-1}+1}^{2^n} e_i\right\} \Big/ \left\|\sum_{i=1}^{2^n} e_i\right\|, \cdots,$$

$$r_n = \left\{\sum_{i=1} (-1)^{i+1} e_i\right\} \Big/ \left\|\sum_{i=1} e_i\right\|$$

satisfy, for $a_1, \cdots, a_n$ arbitrary, $\|\Sigma_{i=1}^{n} a_i r_i\| \leqq \alpha(\Sigma a_i^2)^{1/2}$. Let $f_1, \cdots, f_n$ be positive disjoint elements of Λ_1 and define $T : \Lambda_1 \rightarrow \Lambda_2$ by $Tx = \Sigma f_i(x) r_i$. It is readily checked that for $x \geqq 0$, one has $|T|x \geqq \Sigma f_i(x)|r_i|$ so that $\| |T| x \| \geqq \Sigma f_i(x)$. In particular, we have $\|\Sigma f_i\| \leqq \| \, |T| \, \| \leqq K \|T\|$. On the other hand, $\| Tx \| \leqq \alpha(\Sigma f_i(x)^2)^{1/2}$ for any x, so that

$$\|\Sigma f_i\| \leqq K\alpha \sup_{\|x\| \leqq 1} (\Sigma f_i(x)^2)^{1/2}$$

$$= K\alpha \sup_{\|x\| \leqq 1} ((\Sigma f_i(x) f_i) x)^{1/2} \leqq K\alpha \sup_{\|x\| \leqq 1} \|\Sigma f_i(x) f_i\|^{1/2}$$

$$\leqq K\alpha \sup_{1 \leqq j \leqq n} (\|f_j\| \|\Sigma f_i\|)^{1/2} \quad \text{and hence}$$

$$(*) \qquad \|\Sigma f_i\| \leqq (K\alpha)^2 \sup_{1 \leqq i \leqq n} \|f_i\|.$$

Now renorm Λ_1 by $|||x||| = \sup\{\Sigma \|x_i\| \mid x = \Sigma x_i, |x_i| \wedge |x_j| = 0 \text{ when } i \neq j\}$. Then $(\Lambda_1, |||\cdot|||)$ is an AL - space equivalent to $(\Lambda_1, \|\cdot\|)$ by (*). This completes the proof.

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